The Design of the Bow [1]

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Abstract

The invention of the bow and arrow probably ranks for social impact with the invention of the art of kindling a fire and the invention of the wheel. It must have been in prehistoric times that the first missile was launched with a bow, we do not know where and when. The event may well have occurred in different parts of the world at about the same time or at widely differing times.

Numerous kinds of bows are known, they may have long limbs or short limbs, upper and lower limbs may be equal or unequal in length whilst cross-sections of the limbs may take various shapes. Wood or steel may be used, singly as in 'self' bows, or mixed when different layers are glued together. There are 'composite' bows with layers of several kinds of organic material, wood, sinew and horn, and, in modern forms, layers of wood and synthetic plastics reinforced with glassfibre or carbon. The shape of the bow when relaxed, may be straight or recurved, where the curvature of the parts of the limbs of the unstrung bow is opposite to the way they are flexed to fit the string.

In previous papers we have dealt with the mechanics of the bow and arrow. The main subject of this paper is the design and construction of bows. Nondimensionalization of the problem leads to the introduction of the maximum elastic energy storage capacity per unit of mass as a material constant for strength. At the same time a coefficient is derived which measures the effectiveness of the form of the cross-section of a bending beam.

The assumption that the limbs of a bow are fully loaded implies a relationship between a number of design parameters. It reduces the dimension of the design parameter space. When the strength of materials is taken into account constraints are imposed on a number of design parameters. This defines feasible regions.

Sensitivity analysis gives the influence of a number of important parameters on quality coefficients, one of these is proportional to the amount of available energy and other are the efficiency and the velocity of the arrow at the onset of its flight. These quality coefficients measure the performance of a bow for specific purposes such as hunting, flight shooting and target shooting.

We compare the performance of bows used in the past with that of modern competition bows. It appears that the superb features of man-made materials contribute most to the better performance of the modern bow and not its geometric shape.

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1 Introduction

A number of disciplines is involved in the design of the bow. One, technology, a discipline in which processability, strength, fracture mechanics, fatigue, maintenance and resistance to atmospheric action, are subjects of interest. Bowyers (manufacturers of archery equipment) have to cope with such questions as the influence of temperature changes on the performance of the bow and of course with the availability of materials. In the Middle Ages, when longbows were made in huge quantities in England, the fine-grained yew, *Taxus baccata*, was imported from Spain and it was compulsory to import consignment of Spanish bowstaves with each shipment of Spanish wine.

Two, the working of the bow as a mechanical device can be described by mathematical formulations of physical laws, *e.g.* Newton's law or the principle of Hamilton. The action of the bow before departure of the arrow from the string is called interior ballistics. Exterior ballistics deals with the flight of the arrow through air.

Three, the success of the archer depends largely upon his or her skill. For, when mechanical systems are used to launch arrows, the point of impact is approximately the same for each shot. Small deviations may occur due to differences in the mass of the arrow or wind-velocity/direction. Thus an archer will try to perfect a technique that produces a repeatable shot, and he or she must gain experience in coping with changing weather conditions.

In the past bowyers relied heavily upon experience for the design of the bow. The performance of the bow was improved by the 'try and cut' method. The bowmaker was a craftsman and he made his bows very much as his father had done before him, any improvements being strictly empirical, he did not write books about it — apprentices learned their craft by word of mouth and practical demonstration. In the 1930's bows and arrows became objects of study by scientists and engineers, for example by Hickman, Klopsteg and Nagler, see [5–7].

In previous papers, [8–12] we dealt with the mechanics of the different types of bow: *non-recurve*, *static-recurve* and *working-recurve* bows. The main subject of this paper is the design of the bow. We will formulate the problem in the terminology of the optimization theory for mechanical structures. It is thus clear that the strength of materials used in making a bow, is dealt with. We consider only bows of traditional design, consisting of a bent stick with a grip in the middle. We do not deal with newer developments in which mechanical devices such as pulley's and springs are used to improve the design, see [1,13,14].

The problem is formulated and different types of bow are defined in Section 2. All design parameters are charted accurately and quality coefficients are defined. Besides real number parameters there are three functions of the length coordinate along the limbs, *i.e.* bending stiffness, mass distribution and geometric shape of the unstrung bow, giving rise to an intricate design process. The importance of the application of dimensional analysis will be dealt with.

The mechanics of the bow is dealt with in Section 3. However, we are engaged only in aspects related to the design of the bow and this makes this paper self-contained. For a complete discussion of the mathematical formulation of the problem and the derivation of the numerical techniques the reader is referred to former papers, [8–12]. The mechanical performance of bows belonging to different types, is reported. Each 'standard' bow is supposed to represent a bow used in different parts of the world in the past or the present at archery contests such as the Olympic Games. As to the mechanical performance, these bows are much the same and the differences cannot explain the large variation in shooting performance observed in reality.

The subject of Section 4 is the design and construction of the bow where the technological properties of the materials used to make bows are taken into account and the manner in which these materials are applied in bows is considered. Dimension analysis yields two useful quantities. One is the energy storage capacity per unit of mass of the material. The allowable amount for a particular bow–arrow combination depends on the failure mode (fracture, de-lamination, buckling, fatigue, creep) of the limbs. Two, a dimensionless shape-factor of the cross-section of the limbs. This measures the usage of the material in bending.

It is shown that the type of the bow and the properties of the material used are strongly related. Assumptions with respect to a uniform loading along the limbs yield relationships between the design parameters. This reduces the dimension of the design parameter space. Constraints imposed by technological requirements, such as strength, set bounds in the design parameter space. The differences in the performances of the different types of bow can be explained by these feasible regions.

In Section 5 we perform a sensitivity analysis for a number of design parameters in the feasible regions. One of the most important parameters is the mass of the arrow. In practice there is a constraint either on the mass itself, or on the ratio of the arrow mass and mass of the bow. This leads to the introduction of a nominal draw and nominal weight of a bow. They depend on the type of bow and on the materials used. When the weight and draw of a bow equal these nominal values the velocity of the lightest usable arrow is maximal. The sensitivities appear to be different depending on whether the actual weight is larger or smaller than the nominal weight of the type of bow.

In the literature the importance of a large amount of available energy in the fully drawn bow is always stressed. We also analyse sensitivities with respect to this quantity. It is not a design parameter, however, and we discuss how to interpret these sensitivities. Many authors when describing and discussing the famous Turkish flight bow, assign its good performance to its ears. These ears provide leverage and allow more energy to be stored in the bow. Stiff ears, however, are also heavy and therefore diminish efficiency so that the energy of the arrow differs little from that of other types of bow. In addition to this, we show in this paper that part of the good static performance is counteracted by the large amount of elastic energy in the braced bow. This implies heavier limbs to store this extra useless energy and this also reduces the dynamic performance of the bow.

In Section 6 the performance of a number of ancient and a modern bow are compared. It turns out that the application of better materials and a better usage of these materials contributes most to the improvement of the bow.

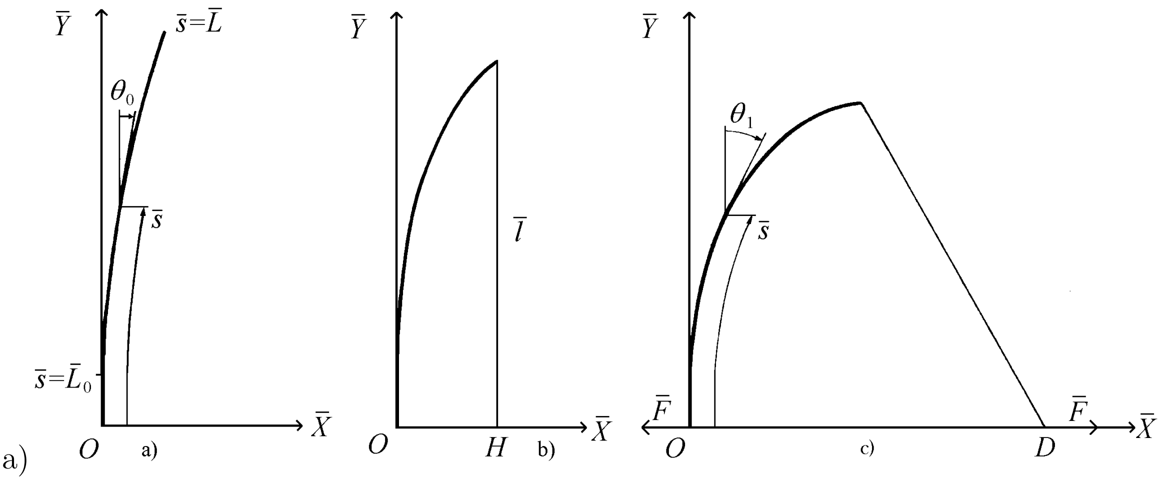


Figure 1: Three situations of the non-recurve bow: a) unbraced, b) braced, c) fully drawn.

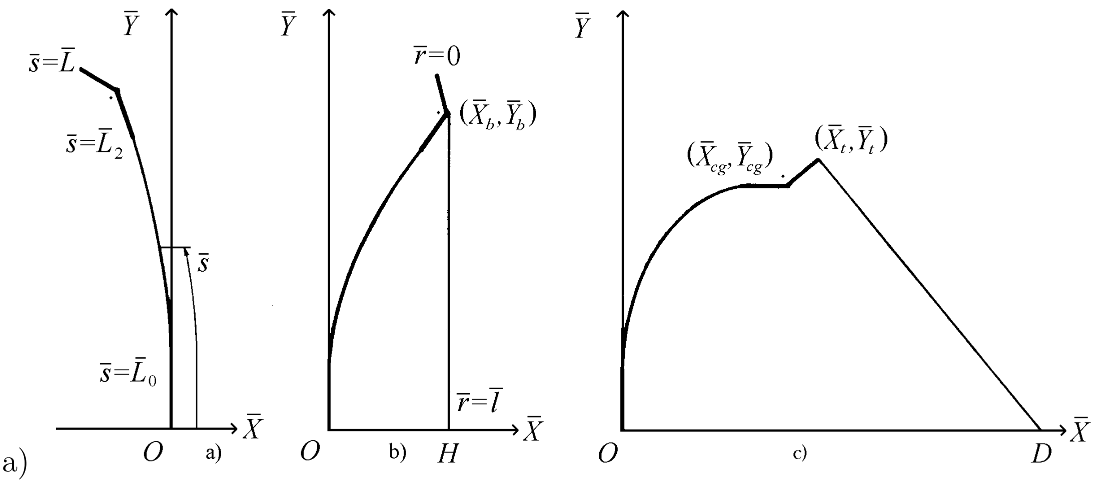
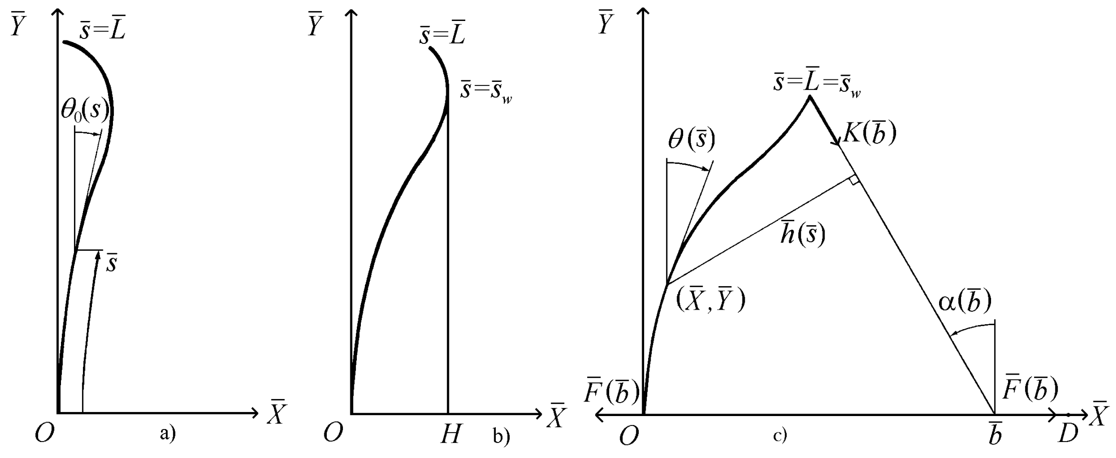


Figure 2: Three situations of the static-recurve bow: a) unbraced, b) braced, c) fully drawn.

2 Classifications of bows

In this paper we deal only with bows that are symmetric with respect to the line of aim. The bow is placed in a Cartesian coordinate system $(\overline{x}, \overline{y})$, the line of symmetry coinciding with the $\overline{x}$-axis and the origin O coinciding with the midpoint of the bow.

We assume the limbs to be inextensible and the Euler-Bernoulli beam theory to be valid. The total length of the bow, measured along it, is denoted as $2\overline{L}$. The bending stiffness $\overline{W}(\overline{s})$ and mass per unit of length $\overline{V}(\overline{s})$ are both functions of the length coordinate $\overline{s}$ measured along the bow from O. In Figure 1(a) the unbraced situation (without string) is shown. The geometry of the bow is described by the local angle $\theta_0(\overline{s})$ between the elastic line and the $\overline{y}$-axis, the subscript 0 indicates the unstrung situation. $\overline{L}_0$ is half the length of the grip and $2\overline{m}_g$ denotes its mass.



a)

Figure 3: Three situations of the working-recurve bow: a) unbraced, b) braced, c) partly drawn.

In Figure 1(b) the bow is braced by applying a string. The length of the unloaded string is denoted as $2\overline{l}_0$, its mass by $2\overline{m}_s$. We assume that the material of the string obeys Hooke's law, longitudinal stiffness is denoted as $\overline{U}_s$. Note that either the length of the string or the brace height, denoted as $|\overline{OH}|$, fixes the shape of the bow in the braced situation.

Classification of the bow is based on geometric shape and the elastic properties of the limbs. The bow of which the upper half is depicted in Figure 1, is called a non-recurve bow. These bows have contact with the string only at their tips ($\overline{s} = \overline{L}$) with coordinates $(\overline{x}_t, \overline{y}_t)$. In the unstrung situation these coordinates are design parameters of the bow and are denoted as $(\overline{x}_{t_0}, \overline{y}_{t_0})$. There may be concentrated masses $\overline{m}_t$ with moment of inertia $\overline{J}_t$ at each of the tips, representing for example horns used to fasten the string. The *simple* straight-end bow and the English longbow, and the *composite* Angular bow are non-recurve bows. Simple bows made out of one piece of wood, straight and tapering towards the ends, have been used by primitives in Africa, South America and Melanesia. Ancient composite bows, *e.g.* the Angular bow used in Egypt and Assyria, consisted of layers of several kinds of organic materials, wood, sinew and horn.

In the static-recurve bow, see Figure 2, the outermost parts of the limbs are stiff. These parts are called *ears*. The mass of each ear and its moment of inertia with respect to the centre of gravity $(\overline{x}_{cg}, \overline{y}_{cg})$ are denoted as $\overline{m}_e$ and $\overline{J}_e$, respectively. The flexible part $\overline{L}_0 \leq \overline{s} \leq \overline{L}_2$ is called the *working part* of the limb. Some Tartar, Chinese, Persian, Indian and Turkish composite bows are static-recurve bows. In the braced situation the string rests upon string-bridges, see Figure 2(b). These string-bridges are fitted to prevent the string from slipping past the limbs. The position of the bridge on the upper limb is referred to as $(\overline{x}_b, \overline{y}_b)$. When such a bow is drawn, at some moment the string leaves the bridges and has contact with the limbs only at the tips.

With a working-recurve bow the parts near the tips are elastic and bend during the final part of the draw, see Figure 3. When drawing such a bow the length of contact between string and limb decreases gradually until the point where the string leaves the limb, denoted as $\overline{s} = \overline{s}_w$, coincides with the tip $\overline{s} = \overline{L}$. It remains there during the final part of the draw. We assume that there is no friction between bow and string for $\overline{s}_w \leq \overline{s} \leq \overline{L}$. Today almost all bows seen at target archery events, are working-recurve bows. The core of these modern composite bows is made of wood, for instance Maple, with layers of synthetic plastics reinforced with glassfibre or carbon on both sides. In Figure 3(c) the bow is pulled by force $\overline{F}(\overline{b})$ into a partly drawn position where the middle of the string has the $\overline{x}$-coordinate $\overline{b}$. To each bow belongs a value $\overline{b} = \overline{b}_1 = |\overline{OD}|$ for which it is called fully drawn. Values of variables in this situation are indicated by a subscript 1. The force $\overline{F}(|\overline{OD}|)$ is called the *weight* of the bow and the distance $|\overline{OD}|$ is its *draw*. The arrow, represented by a point mass $2\overline{m}_a$, is propelled towards its target by releasing the drawn string at time $\overline{t} = 0$ and holding the grip of the bow at its place. During acceleration at some specific point in time $\overline{t} = \overline{t}_b$ the string again touches the belly side of the limb of a static-recurve or working-recurve bow.

In practice, the connection between the end or nock of the arrow and the string is not loose as is assumed usually. It is stated that the strength of this connection, called the nock tension, should be such that the arrow will just hang on the bow string without falling off. We assume that this nock-tension is zero, then the arrow leaves the string when the acceleration of the midpoint of the string becomes negative. This moment is denoted as $\overline{t}_l$ and its velocity at that moment is called *muzzle velocity* or *initial velocity* of the arrow which is denoted as $\overline{c}_l$. The arrow in its flight is slowed down by the drag or resistance of the air. The exterior ballistics is beyond the scope of this paper. We do, however, deal with the vibratory motion of the bow limbs after arrow exit.

A shorthand notation for a bow and arrow combination is introduced as follows

$$\begin{aligned}\overline{B}(\overline{L}, \overline{L}_0, \overline{W}(\overline{s}), \overline{V}(\overline{s}), \theta_0(\overline{s}), \overline{m}_a, \overline{m}_t, \overline{J}_t, \overline{m}_e, \overline{J}_e, \overline{m}_g, \overline{x}_{cg_0}, \overline{y}_{cg_0}, \overline{x}_{b_0}, \overline{y}_{b_0}, \overline{x}_{t_0}, \overline{y}_{t_0}, \overline{L}_2,\\ \overline{U}_s, \overline{m}_s, |\overline{OH}| \text{ or } \overline{l}_0; |\overline{OD}|, \overline{F}(|\overline{OD}|), \overline{m}_b),\end{aligned} \tag{1}$$

where $\overline{m}_b$ is the mass of one limb excluding the mass of the grip and including the mass of the ears.

The quantities $|\overline{OD}|$, $\overline{F}(|\overline{OD}|)$ and $\overline{m}_b$ are taken as the elements of a dimensional base in a dimensional analysis. The first factor is limited by the length of the bowman's arms, from the left hand fully extended to the right hand, drawn back beside the right shoulder. The second one is linked up with the bowman's muscle. In addition it depends on the ability of the fingers of the right hand to control the string, to hold it during aiming and release it at the right moment. The third factor is a limitation of the strength of the materials used. Quantities with dimension are labelled by means of a bar '—' and quantities without the bar are the associated dimensionless quantities.

Note that the two latter parameters the weight of the bow $\overline{F}(|\overline{OD}|)$ and the mass of the limbs $\overline{m}_b$ have been added to the list artificially. This implies that both functions $W(s)$ and $V(s)$ are constrained. Both functions, bending stiffness $\overline{W}(\overline{s})$ and mass distribution $\overline{V}(\overline{s})$ are considered to be the product of the functions $\overline{W}(\overline{s})/\overline{W}(\overline{L}_0)$ and $\overline{V}(\overline{s})/\overline{V}(\overline{L}_0)$ of the length coordinate $\overline{s}$ into $\mathbf{R}$ and parameters $\overline{W}(\overline{L}_0)$ and $\overline{V}(\overline{L}_0)$ with dimensions. These resulting functions together with the function $\theta_0(\overline{s})$ make the bow a distributed parameter structure. Thus the values of the functions $W(s)$ and $V(s)$ for $s = L_0$ are already fixed by two constraints. The first constraint relating to weight, is an implicit relationship between a number of parameters, of which $W(s)$ is one, and the weight $F(|OD|) = 1$ of the bow. The second constraint is

$$m_b = 1 = \int_{L_0}^{L_2} V(s)ds + m_e \ . \tag{2}$$

2.1 Quality coefficients

The purpose for which the bow is used has to be considered in the definition of a cost-function that should be optimized to obtain the 'best' bow. It appears, however, to be very difficult to define such a cost-function uniquely. Therefore we introduce a number of quality coefficients which can be used to judge the performance of a bow and arrow combination.

In the past the bow was used for hunting and warfare requiring a good arrow penetration capacity. High arrow kinetic energy at the moment of impact and perhaps also a large linear momentum are indispensable. Initial velocity must be high enough to ensure a flat trajectory of the arrow. Nowadays archery has become a sport, for target shooting accuracy is most important. This means that the action of the bow should not exaggerate small differences in the parameters caused by slight differences in the handling of the bow by the archer during the shooting.

In flight shooting, it is tried to shoot an arrow as far as possible. When the drag during the flight of the arrow is neglected, for maximum range its initial velocity must be as large as possible and the angle of departure must be 45° with the horizontal.

We will now define three dimensionless quality coefficients. First, the static quality coefficient q is given by

$$q = \frac{\overline{A}}{|\overline{OD}|\,\overline{F}(|\overline{OD}|)} \quad \text{and} \quad \overline{A} = \int_{\overline{b}=|\overline{OH}|}^{\overline{b}=|\overline{OD}|} \overline{F}(\overline{b})d\overline{b} \ , \tag{3}$$

where $\overline{A}$ is the energy stored in the elastic parts of the bow, the working parts of the limb and the string, by deforming the bow from the braced position into the fully drawn position. Hence, $\overline{A}$ measures the total amount of energy available for the acceleration of the arrow. The static quality coefficient is large when the shape of the Static Force Draw curve $\overline{F}(\overline{b})$ is a concave function. It is small when the drawing force rises very sharply in the last part of the draw. In this case, while aiming, the archer has to hold a relatively large force to obtain a certain amount of available energy. Furthermore the available energy depends more sensitively on the actual draw length. Notice that q can be larger than 1 but is generally about 0.5.

Second, the shooting efficiency η defined by

$$\eta = \frac{\overline{m}_a \, \overline{c}_l^2}{\overline{A}} \ , \tag{4}$$

where $\overline{c}_l$ is again the initial velocity, is a dynamic quality coefficient. Using the principle of stationary potential energy we can show that under general conditions we have $\eta \leq 1$. When the braced situation is the only stable equilibrium the potential energy possesses an unique minimum. Therefore the potential energy in the bow at arrow exit is larger than or equal to this minimum and as a result the difference with the potential energy in the fully drawn position is generally smaller for the shape at arrow exit than it is in the braced situation. Thus even when the velocity of the limb is zero we generally have $\eta \leq 1$. When there are more than one local minima for the potential energy in the braced situation we could have $\eta \geq 1$. In [12] we dealt with this phenomenon. A bow with a very flexible part of the limbs near the tip had three braced shapes, two stable and an unstable equilibrium of the bow without an external force. Drawing from the largest of the two local minima, the shape of the bow at arrow exit could be close to the shape of the braced bow belonging to the smallest minimum of the potential energy, the global minimum. Before the next shot, the archer has to put the bow in the other stable braced situation again before drawing the bow. This extra amount of energy is not taken into account in our definition of the efficiency, but it is available for the acceleration of the arrow.

In [12] we showed that theoretically a shooting efficiency of 100% can be obtained if the limbs of the bow are taken to be rigid with all the elasticity to be concentrated in two elastic hinges and an inextensible string without mass. This simple model shows the principle of the bow very clearly, because of geometrical constraints the velocity of the relatively heavy limbs at arrow exit is small, while the very light string is connected to the fast moving arrow. This implies that the kinetic energy of the moving parts of the bow at arrow exit is relative small and therefore almost all available energy is transferred, as kinetic energy, to the high speed light arrow.

The initial velocity of the arrow then follows from Equations (3) and (4), being

$$\overline{c}_l = \sqrt{\frac{q\eta}{m_a} \, \frac{|\overline{OD}| \, \overline{F}(|\overline{OD}|)}{\overline{m}_b}} = \nu \sqrt{\overline{d}_{bv}} \; , \tag{5}$$

where we introduced a factor $\overline{d}_{bv}$ and the dimensionless initial velocity referred to as ν according to

$$\overline{d}_{bv} = \frac{|\overline{OD}| \, \overline{F}(|\overline{OD}|)}{\overline{m}_b} \quad \text{and} \quad \nu = \sqrt{\frac{q\eta}{m_a}} \; , \tag{6}$$

respectively. This dimensionless initial velocity ν is taken as the third quality coefficient. This is determined by the other two coefficients, q Equation (3), η Equation (4) and the dimensionless mass of the arrow m_a.

The efficiency η depends highly on the mass of the string. This influence can be assessed using a simplified model in which we assume that the string remains straight between the points of attachment at the tip of the bow and the nock of the arrow. The resulting distribution of the kinetic energy along the string indicates that 1/3 of the mass of the string should be added to the arrow after which the string can be taken without mass.

Then,

$$\max(\eta) \approx \frac{m_a}{m_a + \frac{1}{3}\, m_s} \ , \tag{7}$$

gives an approximation of the maximum attainable efficiency. The reason is that for the unrealistic simple model with the two rigid limbs described above, but now adding a string with mass moving in the manner assumed above, the efficiency would be equal to the right-hand side of Equation (7).

Klopsteg improved Equation (7) by stating that a certain part of the deformation energy $\overline{A}$ is converted into kinetic energy of the arrow, the string and the bow limbs. He suggested the formula

$$\overline{A} = (\overline{m}_a + \overline{K}_h)\, \overline{c}_l^{\,2} \ , \tag{8}$$

where $2\overline{K}_h$ is the *virtual mass*. This quantity can be estimated from measured velocities for different arrow masses or can be calculated using a mathematical model. However, it accounts as we see later on not only for the moving limbs, but also for the elastic energy in the limbs and the string, exceeding the value of the elastic energy in the braced situation referred to as $\overline{A}_H$. Klopsteg states: "That the virtual mass is in fact a constant, has been determined in many measurements with a large number of bow". Experiments described in [18] with a modern working-recurve bow confirmed Klopsteg's rule very well. In the next sections we will therefore use the following expression for efficiency

$$\eta = \frac{m_a}{m_a + K_h} \ , \tag{9}$$

where K_h is the dimensionless half virtual mass. The quantity K_h is, just as is the efficiency η, a function of dimensionless parameters, however, it does not depend on m_a. This rule is not based on a physical law, we found in [9] that it was not supported by the results obtained with our mathematical model for a straight-end bow.

Observe that by definition the quality coefficients q, η and ν mentioned above are dimensionless. This means that the quality coefficients measure the quality of the bow and arrow combination under comparable conditions; the same weight, draw and mass of the limbs.

In an input–output philosophy all parameters in Equation (1) are input variables which determine the mechanical action of the bow and arrow combination. In this notation $\overline{B}$ stands for the set of output variables in which one is interested, for instance the quality coefficients.

3 The mechanics of the bow and arrow

In our mathematical model the action of a bow and arrow is fixed by one point in a 24 dimensional parameter space, of which 3 parameters are functions of $\overline{s}$ into $\mathbf{R}$, see

Equation (1). In what follows we study a number of standard bows each representing different sorts of bow. These different sorts used in the past and at present, form clusters around these standard bows in the parameter space. We consider the three types of bow: two non-recurve bows, namely the Straight-end bow (for instance the Longbow) and the Angular bow; two Asian static-recurve bows and two working-recurve bows, one with an extreme recurve and a modern working-recurve bow. The static deformation shapes of all these bows are shown in [10].

For the functions $W, V, \theta : [L_0, L] \to \mathbf{R}$ we assume a simple form:

$$\begin{aligned} \mathcal{W}(s) &= \mathcal{W}(L_0)\,\frac{L-s}{L-L_0}\ ,\ L_0 \le s \le L-\epsilon(L-L_0)\ , \\ \mathcal{W}(s) &= \epsilon\mathcal{W}(L_0)\ ,\ L-\epsilon(L-L_0) \le s \le L\ , \end{aligned} \tag{10}$$

where $\mathcal{W}$ is W or V. If not stated otherwise the shape of the unstrung bow is given by

$$\theta_0(s) = \theta_0(L_0) + \kappa_0\,\frac{L-s}{L-L_0}\ ,\ L_0 \le s \le L\ . \tag{11}$$

The two parameters $W(L_0)$ and $V(L_0)$ are already fixed by the requirement that $|OD|$, $F(|OD|)$ and m_b are equal to 1. Under this description, these functions are fixed by only three parameters ϵ, $\theta_0(L_0)$, a measure for the reflex of the grip, and κ_0, a measure for the recurve of the unstrung limbs in a circular arc.

We start with a straight-end bow described by Klopsteg in [5]. This bow is referred to as the KL-bow and it represents among others the simple bows used by primitives all over the world and the famous English longbow.

The values for all the parameters are given in Table 1. W, V and θ are given in Equation (10) with $\epsilon = 1/3$ and Equation (11) with $\theta_0(L_0) = 0$ and $\kappa_0 = 0$.

The AN-bow represents the Angular bow found in Egypt and Assyria. For this bow the bending stiffness and mass distribution $W(s)$ and $V(s)$ are given in Equation (10) with $\epsilon = 1/3$ and $L_0 = 0$, while $\theta_0(s)$ is given by

$$\begin{aligned} \theta_0(s) &= \arcsin\Big(\frac{|OH|}{L}\Big) - \frac{s}{L}\ ,\ 0 \le s \le L-\epsilon L\ , \\ \theta_0(s) &= \arcsin\Big(\frac{|OH|}{L}\Big) - \Big[\frac{s}{L} - \frac{1}{2\epsilon}\Big(\epsilon - \frac{L-s}{L}\Big)^2\Big]\ ,\ L-\epsilon L \le s \le L\ . \end{aligned} \tag{12}$$

This function gives, in the braced position, the characteristic triangular shape.

Now we consider two static-recurve bows found in China, India or Persia, called the PE-bow, and one from Turkey, called the TU bow. The bending stiffness and mass distribution of these two bows $W(s)$, $V(s)$ and the shape of the unstrung bow, $\theta_0(s)$ are given in Equation (10) with $\epsilon = 1/3$ and Equation (11) for $L_0 \le s \le L_2$ with $\theta_0(L_0) = 0$ and $\kappa_0 = 0$. For the TU-bow all the parameters are equal to those of the PE-bow except $\kappa_0 = -2.0$ and adapted values for $x_{cg_0}, y_{cg_0}, x_{b_0}, y_{b_0}, x_{t_0}, y_{t_0}$ so that the ears equal those of the PE-bow are placed in line with the working part of the limbs at $s = L_2$.

Table 1: The complete set of design parameters for the standard bows studied. The parameters used to make the parameters dimensionless are: $|\overline{OD}|$, $\overline{F}(|\overline{OD}|)$ and $\overline{m}_b$. For all bows we have: $m_t = 0$, $J_t = 0$ and $m_g = 0$.

Bow	KL	AN	PE	ER
L	1.286	0.7857	1.0	1.151
L_0	0.1429	0	0.1429	0.2075
$W(s)$, ϵ in Eqn. (10)	1/3	1/3	1/3	1/32
$V(s)$, ϵ in Eqn. (10)	1/3	1/3	1/3	1/32
$\theta(s)$	Eqn. (11)	Eqn. (12)	Eqn. (11)	Eqn. (11)
m_a	0.0769	0.0769	0.0769	0.0769
m_e	0	0	0.4	0
J_e	0	0	0.0025	0
x_{cg_0}	—	—	-0.0464	—
y_{cg_0}	—	—	0.7589	—
x_{b_0}	—	—	0	—
y_{b_0}	—	—	0.7857	—
x_{t_0}	0	-0.1551	-0.1856	-0.4153
y_{t_0}	1.286	0.7427	0.8929	0.5824
L_2	—	—	0.5714	—
U_s	131	131	131	131
m_s	0.0209	0.0209	0.0209	0.0209
$\|OH\|$	0.214	0.214	0.25	0.214

One example of a working-recurve bow possesses an excessive recurve and is called the ER-bow. It resembles a bow described and shot by Hickman, see [5]. The bending stiffness distribution is given by Equation (10) with $\epsilon = 1/32$, while $\theta_0(s)$ given by Equation (11) with $\theta_0(L_0) = 0.8404$ and $\kappa_0 = -3.3537$.

The energy values for different parts of these bows in various situations are shown in Table 2. These values indicate that for bows with strongly recurved limbs, the amount of energy in the braced situation (referred to as A_H) is high (for the KL-bow $A_H = 0.1226$ while for the TU-bow we calculated $A_H = 1.0867$ and for the ER-bow even $A_H = 1.4260$). At arrow exit the stored potential energy is, for all bows, higher than in the braced situation (for instance for the AN-bow 0.1765 and 0.1576, respectively). This shows that K_h accounts also for the excess of the elastic energy and that it is not solely an added mass accounting for the kinetic energy of the limb and string. The amount of energy in fully drawn situation, stored in the elastic parts of the bow is referred to as A_D. It is equal to $A_b + A_s$, where A_b is the amount of elastic energy in the limbs and A_s in the string, of the fully drawn bow. For example for the ER-bow we have $A_D = 1.415 + 0.012 = 1.426$.

The three calculated quality coefficients for these types of bow are shown in Table 3. The static quality coefficient q is given by: $q = A_D - A_H$ with A_D and A_H given in Table 2. For example for the PE-bow we have $q = 0.5932 - 0.1625 = 0.432$. The quantity A_b, the elastic energy in the fully drawn limbs (already given in Table 2) is also given because this quantity will play an important role in the design of the bow dealt with in Section 4. The efficiency of the strongly recurved ER-bow is rather bad $\eta = 0.810/0.3374 = 0.417$ and this undoubtedly counteracts the very good static quality coefficient $q = .810$. Obviously, the tips of the limbs are decelerated less efficiently.

The values of the quality coefficients for a modern working-recurve bow are also given in Table 3. This WR-bow is described in [8]. The dimensionless mass of the arrow and string of this bow were respectively $m_a = 0.0629$ and $m_s = 0.0222$. With these masses the calculated quality coefficients are given in [11]: the efficiency is $\eta = 0.729$ and the initial velocity $\nu = 2.23$. The values for the efficiency η and initial velocity ν shown in Table 3 were adjusted for the values $m_a = 0.0769$ and $m_s = 0.0209$ using Klopsteg's rule (9) taking (7) into account by the assumption that the term $1/3\, m_s$ is part of the virtual mass K_h.

The results indicate that the initial velocity ν is about the same for all types! So, within certain limits, the dimensionless parameters which determine the mechanical performance of the bow for flight shooting, appear to be less important than is often claimed.

The acceleration force E acting on the arrow plus the added mass of the string $1/3\, m_s$ is defined by

$$E = -2\,(m_a + \frac{1}{3} m_s)\dot{c}\ ,\ t \geq 0\ . \tag{13}$$

where $c = \dot{b}$ is the dimensionless velocity of the arrow. Another important quantity is the recoil force P, being the force the bow exerts on the bowhand of the archer.

These forces E, P and the force in the string K as a function of time t are shown in Figure 4. After arrow exit at time $t = t_l$ in Figure 4 the acceleration force E is the force acting on the added mass representing the string. As can be seen E oscillates around zero

Table 2: Energy distribution for a number of bows in three situations: braced, fully drawn and arrow exit

energy	limbs pot.	kin.	string pot.	kin.	total bow pot.	kin.	arrow kin.
KL-bow							
braced	0.0950	0	0.0276	0	0.1226	0	0
fully drawn	0.5155	0	0.0137	0	0.5292	0	0
arrow exit	0.0663	0.0491	0.0681	0.0344	0.1344	0.0835	0.3112
AN-bow							
braced	0.1461	0	0.0115	0	0.1576	0	0
fully drawn	0.5493	0	0.0033	0	0.5526	0	0
arrow exit	0.1380	0.0573	0.0385	0.0359	0.1765	0.0932	0.2961
PE-bow							
braced	0.1515	0	0.0100	0	0.1625	0	0
fully drawn	0.5879	0	0.0053	0	0.5932	0	0
arrow exit	0.1271	0.0980	0.0506	0.0279	0.1777	0.1271	0.2884
TU-bow							
braced	0.5755	0	0.0204	0	0.5959	0	0
fully drawn	1.081	0	0.0053	0	1.0867	0	0
arrow exit	0.5324	0.1346	0.0844	0.0312	0.6168	0.1658	0.3040
ER-bow							
braced	0.419	0	0.197	0	0.6160	0	0
fully drawn	1.415	0	0.012	0	1.4260	0	0
arrow exit	0.564	0.3511	0.087	0.0756	0.6510	0.4266	0.3374

Table 3: Dimensionless quality coefficients for the standard bows. The dimensionless half mass of the arrow and string of the modern competition WR-bow are $m_a = 0.0629$ and $m_s = 0.0222$. The efficiency η and initial velocity ν of the WR-bow have been adjusted for the reported masses using Klopsteg's rule. The stiffness of its string is about twice those of the other bows.

Bow	q	η	ν	m_a	m_s	$W(L_0)$	$V(L_0)$	A_b
KL-bow	.407	.765	2.01	.0769	.0209	1.4090	1.575	0.5155
AN-bow	.395	.716	1.92	.0769	.0209	0.2385	2.300	0.5493
PE-bow	.432	.668	1.94	.0769	.0209	0.2304	1.867	0.5879
TU-bow	.491	.619	1.99	.0769	.0209	0.1259	1.867	1.0817
ER-bow	.810	.417	2.08	.0769	.0209	0.3015	2.120	1.4150
WR-bow	.434	.770	2.09	.0769	.0209	2.5800	1.950	0.6930

for $t \geq t_l$. The maximum force in the string occurs after arrow exit and it is larger than in the fully drawn situation. This explains why flight shooters let their string break after each shot. A weaker and therefore lighter string yields a higher efficiency η and initial velocity ν.

The results of Tables 2 and 3 show that the modern working-recurve bow WR is a good compromise between the non-recurve bow and the static-recurve bow. The recurve yields a good static quality coefficient q and the light tips of the limbs give a high efficiency η.

Figure 5 shows the bending moment M as a function of the length coordinate $s - L_0$ for successive times before (Figure 5(a)) and after (Figure 5(b)) arrow exit for the WR-bow. The broken line represents the distribution along the limb in the static braced situation. The static bending moment $M(s)$ is the force in the string K times the distance $h(s)$ as shown in Figure 3(c). This static bending moment is zero when the string lies against the limb near the tip. The distribution along the elastic part of the limb $M_1(s - L_0)$ in the fully drawn bow is also shown in Figure 5(a) and (b).

It follows from Figure 5 that $M(s,t)$ is smaller than $M_1(s)$ except for the part of the limb near the grip after the arrow has left the bow. The differences are small, so we assume that the maximum loading bending moment occurs in the fully drawn situation and it equals $M_1(s)$. In reality there is always internal and external damping which forces the bow and string to return to the braced situation. This reduces the actual bending moments for large t.

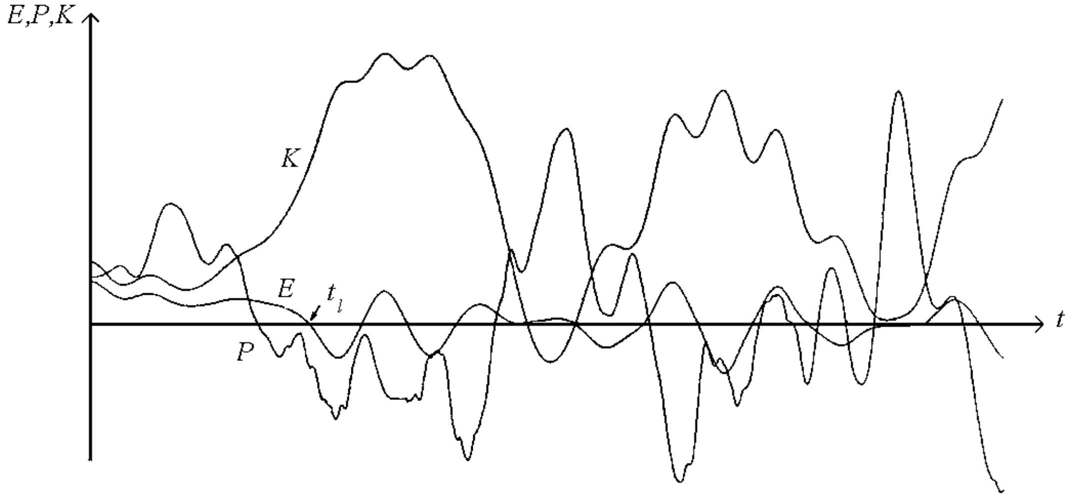


Figure 4: Acceleration force E, string force K and recoil force P for the WR-bow. Moment of arrow exit is $t = t_l$.

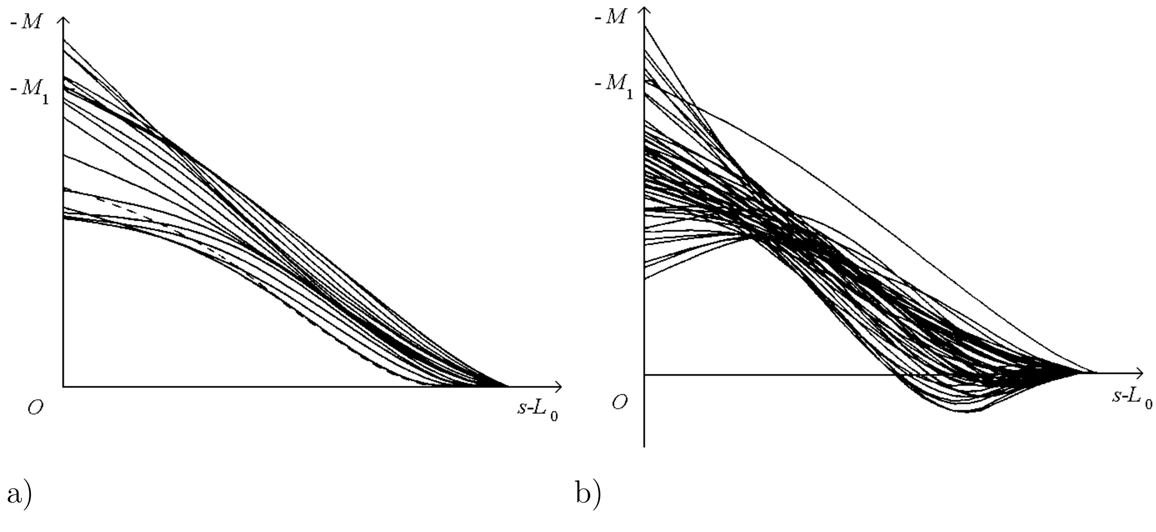


Figure 5: Distribution of the bending moment along the limb of the WR-bow (a) before arrow exit $t \leq t_l$ and (b) after arrow exit $t \geq t_l$. The broken line represents the distribution along the limb in the static braced situation. The bending moment $M_1(s - L_0)$ in the fully drawn bow is also shown in both figures.

4 The design of the bow

The limbs of the bow can be considered to be a beam of variable cross-section $\overline{D}(\overline{s})$ made out of one material with density $\overline{\rho}$ and Young's modulus $\overline{E}$. Assumption of the Euler-Bernoulli hypothesis yields the following relationship between the bending moment $\overline{M}(\overline{s},\overline{t})$ and the normal stress $\overline{\sigma}(\overline{s},\overline{t},\overline{r})$, with $\overline{r}$ the distance from the neutral axis which passes through the centroid of the cross-section

$$\overline{\sigma}(\overline{s},\overline{t},\overline{r}) = \frac{\overline{M}(\overline{s},\overline{t})\,\overline{r}}{\int_{\overline{D}(\overline{s})} \overline{r}^2\, d\overline{D}}\ , \tag{14}$$

where we neglected the influence of the normal and shear forces in the limbs.

In the previous section we showed that it is acceptable to assume that the maximum bending moment for each $\overline{s}$ as a function of time $\overline{t}$ occurs in the fully drawn situation. Then we have, with $\overline{A}_b$ again the elastic energy in the limbs of the bow in fully drawn situation,

$$\frac{\overline{A}_b}{\overline{m}_b} = \frac{2\int_{\overline{L}_0}^{\overline{L}} \int_{\overline{D}(\overline{s})} \frac{1}{2}\frac{\overline{\sigma}_1^2(\overline{s},\overline{r})}{\overline{E}}\, d\overline{D}\, d\overline{s}}{\int_{\overline{L}_0}^{\overline{L}} \overline{\rho}\,\overline{C}(\overline{s})\, d\overline{s}}\ , \tag{15}$$

where $\overline{C}(\overline{s}) = \int_{\overline{D}(\overline{s})} d\overline{D}$ is the area of the cross-section. Thus $\overline{\rho}\,\overline{C}(\overline{s}) = \overline{V}(\overline{s})$ is the mass distribution along the limb. The stress $\overline{\sigma}_1$ is the resulting normal stress due to the bending moment in the fully drawn bow, indicated by the subscript 1.

Two additional useful quality coefficients are defined by

$$\overline{\delta}_{bv} = \frac{\overline{\sigma}_w^2}{2\,\overline{\rho}\overline{E}} \quad , \quad a_D = \frac{\int_{\overline{L}_0}^{\overline{L}} \int_{\overline{D}(\overline{s})} \frac{1}{2}\overline{\sigma}_1^2(\overline{s},\overline{r})\, d\overline{D}\, d\overline{s}}{\frac{1}{2}\overline{\sigma}_w^2 \int_{\overline{L}_0}^{\overline{L}} \overline{C}(\overline{s})\, d\overline{s}}\ , \tag{16}$$

where $\overline{\sigma}_w$ is the working stress of the material, equal to the yield point or the ultimate strength divided by a factor of safety and the quantity $\overline{\delta}_{bv}$ is the amount of energy which can be stored per unit of mass in the material. Then

$$\frac{\overline{A}_b}{\overline{m}_b} = 2a_D\,\overline{\delta}_{bv}\ . \tag{17}$$

In Table 4 an estimation of $\overline{\delta}_{bv}$ for some materials used in making bows, is given.

The maximum value for the dimensionless coefficient a_D equals 1 in the homogeneous stress-state $\overline{\sigma}_1^2(\overline{s},\overline{r}) = \overline{\sigma}_w^2$ for all $\overline{s}$ and $\overline{r}$. However, a_D is generally smaller than 1 for two reasons.

First, the stress in the fibres near the neutral axis is smaller than the working stress and this reduces the coefficient a_D. Suppose the stress in the outermost fibres equals the

Table 4: Mechanical properties and the energy per unit of mass $\overline{\delta}_{bv}$ for some materials used in making bows, see also [3].

material	$\overline{\sigma}_w$ N/m^2 10^7	$\overline{E}$ N/m^2 10^{10}	$\overline{\rho}$ kg/m^3	$\overline{\delta}_{bv}$ Nm/kg
steel	70.0	21.0	7800	130
sinew	7.0	.09	1100	2500
horn	9.0	.22	1200	1500
yew	12.0	1.0	600	1100
maple	10.8	1.2	700	700
glassfibre	78.5	3.9	1830	4300

working stress for all $\overline{s}$, in this case the limb is called *uniformly stressed*, then we have for each cross-section due to bending

$$\overline{\sigma}_1(\overline{s},\overline{r}) = \overline{\sigma}_w \frac{\overline{r}}{\overline{e}(\overline{s})} , \tag{18}$$

where $\overline{e}$ is the largest of the two distances between the outermost fibres and the neutral axis. For a uniformly stressed bow, a_D is maximum for the given $\overline{D}(\overline{s})$. If the material has the same strength in tension and compression, it will be logical to choose shapes of cross-section in which the centroid is at the middle of the thickness of the limb, equal to $2\overline{e}(\overline{s})$. Then

$$a_D = \frac{\int_{\overline{L}_0}^{\overline{L}} \frac{\overline{I}(\overline{s})}{\overline{e}(\overline{s})^2}\, d\overline{s}}{\int_{\overline{L}_0}^{\overline{L}} \overline{C}(\overline{s})\, d\overline{s}} , \tag{19}$$

where $\overline{I}(\overline{s})$ is the moment of inertia of the cross-section with respect to the neutral axis. The quantity $\overline{EI}(\overline{s}) = \overline{W}(\overline{s})$, already introduced in Section 2, is the distribution of the bending stiffness along the limb. The quantity $2\overline{I}/\overline{e}$ is called the *section modulus*. The magnitude of the moment of inertia and the section modulus are tabulated for various profile sections in commercial use in various handbooks. For the assessment of the shape of the cross-sections we stay with the dimensionless coefficient a_D which follows in a straightforward manner from our statement of the problem.

We will now consider the function $\alpha_D(\overline{s})$ defined by

$$\alpha_D(\overline{s}) = \frac{\overline{I}(\overline{s})}{\overline{C}(\overline{s})\ \overline{e}(\overline{s})^2} , \ \overline{L}_0 \le \overline{s} \le \overline{L} , \tag{20}$$

Table 5: Values of α_D for various shapes of cross-section of limbs

shape of the cross-section D	α_D
rectangular	0.333
elliptical	0.250
D-shaped	0.255
elliptical ring; $1/4(1+k^2)$, $k = 0.9$	0.453

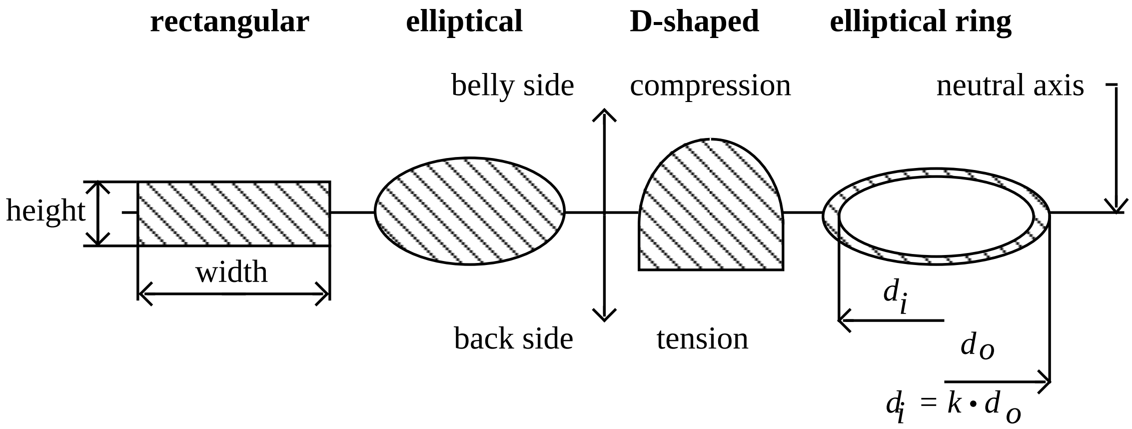


Figure 6: Different shapes of the cross-section of the bow.

for various shapes of cross-section of limbs. For a bow with geometrically similar cross-sections at different values of $\overline{s}$, $\alpha_D(\overline{s})$ is uniform and $a_D = \alpha_D(\overline{s})$. In that case the ratio of similarity with respect to the shape for $\overline{s} = \overline{L}_0$ is a function of $\overline{s}$. Generally, this function is smaller than 1 for $\overline{L}_0 < \overline{s} \leq \overline{L}$.

In Table 5 values of α_D are given for the rectangular, elliptical, D-shaped and elliptical ring cross-sections shown in Figure 6. The English longbow possessed a D-shape cross-section, the belly side approximately formed by a semi-circle, the other side being a rectangle. When the radius of the semi-circle equals the half of the thickness of the limb, α_D equals 0.255, hence smaller than 0.333 for a rectangular shape and larger than 0.25 for a elliptical shape. This magnitude for the D-shape may be conservative. As the shape is not symmetric with respect to the neutral axis, the tensile stress at the back remains smaller than the working stress if the same kind of wood is used throughout the limb. The back of such a yew bow, however, was usually made of white sapwood which has a lower Young's modulus than the red heartwood used for the belly side, see [16]. Hence, the neutral axis is moved toward the belly side to compensate for the shift due to the semi-circular shape of the belly. Furthermore experiments showed the sapwood of yew to be stronger than the heartwood and this indicates that this combination of the material used and shape of the cross-section can yield a good design.

Steel bows, for instance the Seefab bow invented in the 1930's, are designed on the principle of a flattened tube. We assume that the inner major axis d_i equals k times the outer major axis d_o of the concentric ellipses. For $k = 0.9$ the value of α_D may become 0.4525, so larger than the other values mentioned. This is to be expected because relatively more material is placed near the outermost fibres.

There is still another technique to increase the value of a_D. In ancient Asiatic bows, horn and sinew, together with wood, were used on the belly and back, respectively. Horn is a superb material for compressive strength and sinew laid in glue has a high tensile strength, see Table 4. The Young's modulus of both materials is rather small, but the permissible strain is very high. This shows that the difference of the curvature of the axis of the limb between fully drawn and unstrung situation has to be large, or the thickness of the limb has to be large to load these materials to the full extent. In the latter case the space between the two materials near the outermost fibres is filled in with light wood, which has to withstand the shearing stresses and keep the horn and sinew sufficiently apart. In modern bows horn and sinew are replaced by synthetic plastics reinforced with glassfibre or carbon. So, in composite bows not only are better materials used, but they are also used in a more profitable manner. For composite bows we define equivalent quantities for the Young's modulus and density so that it has the same mechanical action with respect to bending as the simple bow made out of one kind of material. If these magnitudes are substituted in the product $a_D \overline{\delta}_{bv}$ this product can be substantially larger than for simple wooden bows.

Second, the tensile and compressive stresses in the fully drawn bow in the outermost fibres of the limb, in the back and the belly respectively, may be less than the working stress. For materials with reduced strength in compression and greater strength in tension, as is the case in most woods, the advisable cross-section for the limb will not be symmetrical with respect to the neutral axis, but will be such that the distance from the neutral axis to the most remote fibres in tension and compression are in the same ratio as the working stresses, $\overline{\sigma}_w$, of the material in tension and compression. Note that even for a symmetrical cross-section the neutral axis does not coincide with the line of symmetry when the Young's moduli for tension and compression are not equal. For wood the neutral axis lies closer to the outermost fibres in tension than to the outermost fibres in compression because the modulus in tension is generally larger than in compression. With the design of the limbs one has to assure stability of the limb, without tendency to twist or distort laterally when the bow is drawn. In practice this is accomplished by the requirement that the width of a limb may not become smaller than its thickness, in this case material near to the tips is not used to the full extent.

Table 6: An estimation of maximum thickness of the limbs for a number of types of bow, $|\overline{OD}| = 0.7112$ m.

Material	Bow type	$W(L_0)$	$M_1(L_0)$	$\overline{\sigma}_w$ N/m^2 10^7	$\overline{E}$ N/m^2 10^{10}	$\overline{e}(L_0)$ m	thickness(L_0) m
yew	KL	1.41	1.02	11	1	0.0112	0.022
sinew	PE	0.23	0.64	7	0.09	0.020	0.030
horn	PE	0.23	0.64	9	0.22	0.010	
sinew	TU	0.12	0.64	7	0.09	0.0133	0.019
horn	TU	0.12	0.64	9	0.22	0.0055	
sinew	KL	1.41	1.02	7	0.09	0.0765	0.153
yew	TU	0.12	0.64	11	1	0.0016	0.0032

4.1 The construction of the bow

In practice the manufacturer wants to make a bow with a specified draw $|\overline{OD}|$ and weight $\overline{F}(|\overline{OD}|)$ by the use of available materials with a given Young's modulus $\overline{E}$, density $\overline{\rho}$ and working stress $\overline{\sigma}_w$. The equation which makes $\overline{W}(\overline{s})$ dimensionless reads

$$\overline{EI}(\overline{s}) = W(s)\,|\overline{OD}|^2\,\overline{F}(|\overline{OD}|)\ . \tag{21}$$

Equations (14), (18) and (21) show that the thickness of the limb associated with the distance $\overline{e}$ between the outermost fibres and the neutral axis is determined by

$$\overline{e}(\overline{s}) = \frac{\overline{I}(\overline{s})}{\overline{M}_1(\overline{s})}\,\overline{\sigma}_w = \frac{W(s)}{M_1(s)}\,\frac{\overline{\sigma}_w}{\overline{E}}\,|\overline{OD}|\ . \tag{22}$$

These calculations can be done after selecting the static dimensionless parameters L, L_0, $\theta_0(s)$, x_{cg0}, y_{cg0}, x_{b0}, y_{b0}, x_{t0}, y_{t0}, L_2, U_s and $|OH|$ or l_0. The solution of the governing static equations yields $M_1(s)$. We recall that we assume that the maximum bending moment occurs in the fully drawn situation. In Section 3 we saw that this is allowed for realistic situations. After the selection of the shape of a cross-section of the limbs $\overline{D}(\overline{s})$, the width is determined by Equation (21). To ensure stability this width should not be taken smaller than the thickness of the limb. There after the area of the cross-section $\overline{C}(\overline{s})$ can be calculated. $\overline{C}(\overline{s})$ and the density $\overline{\rho}$ jointly determine the mass per unit length $\overline{V}(\overline{s})$ and with the length of the limbs $\overline{L}$ the mass of the limbs $\overline{m}_b$. This completes the design of the limbs with a_D as large as possible, depending on the chosen $\overline{D}(\overline{s})$.

We now have the information to calculate the dimensionless mass distribution $V(s)$. The equation which makes $\overline{V}(\overline{s})$ dimensionless reads for $\overline{L}_0 \leq \overline{s} \leq \overline{L}$,

$$V(s) = \overline{V}(\overline{s}) \, \frac{|\overline{OD}|}{\overline{m}_b} = \frac{M_1^2(s)}{\alpha_D(s) W(s)} \, \frac{a_D}{A_b} \, . \tag{23}$$

The mass of the limb $\overline{m}_b$ is derived from Equation (17)

$$\overline{m}_b = A_b \, \frac{|\overline{OD}| \, \overline{F}(|\overline{OD}|)}{2 a_D \, \overline{\delta}_{bv}} \, . \tag{24}$$

When the product $a_D \, \overline{\delta}_{bv}$ is known, the three quantities $|\overline{OD}|$, $\overline{F}(|\overline{OD}|)$ and $\overline{m}_b$ of the dimensional base are mutually dependent. This shows that, when technology is taken into account, it would be better to take the quantity $\overline{\delta}_{bv}$ as the third element of the dimensional base instead of the mass of the bow. We adhere to the mass of the limb in order to be consequent with the theory on the mechanics of the bow and arrow, derived in [8–12]. The design and construction of the bow is complete after the remaining parameters m_a, m_t, J_t, m_e, J_e, m_g, m_s are selected.

Hence, in a uniformly stressed bow with $\alpha_D(s) = a_D$ along the limbs, the mass distribution $V(s)$ depends on other dimensionless parameters given in Equation (1), see (Equation 23). Other parameters given in Equation (1) are also mutually dependent in practice, for example the parameters relating to the ears, $\overline{m}_e$, $\overline{J}_e$, $\overline{x}_{cg0}$, $\overline{y}_{cg_0}$, $\overline{x}_{b0}$, $\overline{y}_{b_0}$, $\overline{x}_{t0}$, $\overline{y}_{t_0}$, $\overline{L}_2$ are strongly related.

The same reflections on energy storage capacity can be given for the materials of the string. To reduce the mass of the string strong materials should be used which are able to store large amounts of elastic energy for a given mass. Many kinds of fibre have been used for making strings. In the past natural fibres were used, animal fibres (silk and sinew) and vegetable fibres (hemp, linen, cotton and strips of bamboo or rattan) and Belgian strings made of long-fibered Flemish flax have been famous. Recently, man-made fibres such as Dacron, Kevlar and Twaron were developed. Note that, except for the loop with which it is fastened to the bow, the material throughout the cross-section of the string is uniformly stressed. The maximum force determines, together with the strength of the material, the minimum mass of the string. Dynamic computer simulations, see Figure 4, show that maximum force does not occur in the fully drawn situation. So, there is a relationship between the string parameters, $\overline{U}_s$, $\overline{m}_s$ and the strength of the material used for the string, and the maximum force in the string. This force is not known from static calculations, so an initial guess has to be made, which must be checked after the dynamic calculations.

In practice two string-parameters, the mass $\overline{m}_s$ and the stiffness $\overline{U}_s$, are for a particular string material fixed by the number of strands. An increase of this number, to be denoted as n_s, makes the string stiffer (generally this implies a higher initial velocity) but also heavier (and this counteracts the advantage of a stiffer string). This is a simple example that shows how technological considerations can reduce the dimension of the design parameter space.

We can calculate the thickness of the limbs near the grip ($\overline{s} = \overline{L}_0$) for different types of bow made of the different materials. The distance $\overline{e}$ between the outermost fibres and the neutral axis calculated using Equation (22) is shown in Table 6. The results demonstrate that the yew of the longbow combined in the long straight-end bow and sinew, horn and wood combined in the static-recurve bow, are good combinations, while yew in the static-recurve bow and sinew, horn and wood in the long straight-end bow are not, showing that in practice it is not possible to interchange geometric shape and material in the construction of a bow. We therefore consider only combinations applied generally in the next section.

We will not work out a comprehensive optimization program for the efficiency, initial velocity or amount of kinetic energy in the arrow, in the reduced design parameter space of Equation (1), with constraints for the draw, weight and the maximum amount of energy storage in the material and, possibly, a minimum initial velocity and a minimum mass of the arrow as side conditions. Instead, in the next section we perform a sensitivity analysis within the feasible region for a restricted number of design parameters and in Section 6 we consider a number of different types of bow used in the past and up to present times. We assume that the bow is uniformly stressed in the fully drawn situation, so that $\alpha_D(s) = a_D$ along the limbs. We also assume Klopsteg's rule which makes it possible to uncouple dynamics from statics. Solution of the dynamic behaviour with a single mass of the arrow and string yields an estimation for the parameter K_h which depends solely on static design parameters. Klopsteg's rule Equation (9) then yields the performance of the bow with other arrow and string masses.

5 Sensitivity analysis

The influence of a number of the design parameters on the performance of the bow is dealt with in this section. We will focus on the initial velocity which needs to be as large as possible for flight shooting.

When we neglect the elastic energy stored in the string, the equation for the initial velocity and the kinetic energy of the arrow follow from

$$\overline{c}_l = \sqrt{2\frac{q}{A_b}\,\frac{\eta}{m_a}\,a_D\,\overline{\delta}_{bv}} \quad \text{and} \quad \overline{m}_a\,\overline{c}_l^2 = 2\,\overline{m}_b\,\frac{q}{A_b}\,\eta\,a_D\,\overline{\delta}_{bv}\ , \tag{25}$$

respectively.

Hence, the initial velocity of an arrow depends on the ratio of the static quality coefficient q and the amount of elastic energy stored in the fully drawn limbs, A_b, on the ratio of the efficiency η and the dimensionless mass of the arrow m_a and finally on the product $a_D\,\overline{\delta}_{bv}$. The kinetic energy is linearly dependent on the mass of the bow, $\overline{m}_b$, whereas it is independent of the mass of the arrow.

When the mass of the arrow itself is a design parameter instead of the dimensionless mass of the arrow, it is advantageous to define a coefficient which resembles $\overline{d}_{bv}$ but instead of being based on the mass of the limb $\overline{m}_b$ it is based on the mass of the arrow $\overline{m}_a$

$$\overline{d}_{av} = \frac{|\overline{OD}|\,\overline{F}(|\overline{OD}|)}{\overline{m}_a}\ . \tag{26}$$

Then

$$\overline{c}_l = \sqrt{q\,\eta\,\overline{d}_{av}} \quad \text{and} \quad \overline{m}_a\,\overline{c}_l^2 = q\,\eta\,|\overline{OD}|\,\overline{F}(|\overline{OD}|)\ . \tag{27}$$

When in what follows the influence of one arbitrary parameter is analysed, all other dimensionless parameters, the draw $|\overline{OD}|$, weight $\overline{F}(|\overline{OD}|)$ and maximum energy storage capacity $a_D\,\overline{\delta}_{bv}$ remain unchanged.

5.1 Influence of the mass of the arrow

Equations (25) and (27) suggest that for a large velocity the mass of the arrow m_a and $\overline{m}_a$ should be as small as possible. The partial derivatives of the dimensionless initial velocity ν and of $\overline{c}_l$ with respect to m_a (Equations (5) and (6)), are given by

$$\frac{\partial \nu}{\partial m_a} = \frac{-1/2\,\sqrt{q}}{(m_a + K_h)^{3/2}} \quad \text{and} \quad \frac{\partial \overline{c}_l}{\partial m_a} = \frac{\partial \nu}{\partial m_a}\sqrt{\frac{2}{A_b}\,a_D\,\overline{\delta}_{bv}}\ , \tag{28}$$

respectively, where we used Klopsteg's rule Equation (9). The first one is always negative, so the mass of the arrow should be as small as possible. We used the fact that the virtual mass K_h is independent on the mass of the arrow m_a. Then Equation (27) and (9) for $\lim m_a \downarrow 0$ yield

$$\max(\overline{c}_l) = \sqrt{\frac{q}{K_h}\overline{d}_{bv}} \quad \text{and} \quad \lim_{m_a \downarrow 0}(\overline{m}_a\,\overline{c}_l^2) = 0\ . \tag{29}$$

Klopsteg [5] called this quantity $\max(\overline{c}_l)$ the *figure of merit* of a bow. In the dimensionless formulations we have $\max(\nu) = \sqrt{q/K_h}$. Obviously no arrow shot with this bow could achieve higher velocity than $\max(\overline{c}_l)$ of Equation (29).

Klopsteg's rule is, however, violated for small values of m_a. For small arrow masses the acceleration force will already be negative before the bow is back in its braced situation. Furthermore, our assumption that the maximum bending moment equals the value in the fully drawn situation, $M_1(s)$, is then obviously violated. Every archer knows that it is not correct to release a fully drawn bow without an arrow. In this case efficiency equals 0, see Equation (29), and the bow or string may break. This shows that there is a minimum value for m_a. This value will depend on the type of bow represented by the other dimensionless parameters of Equation (1) and this relationship is complex. We assume that this minimum, denoted as $\mathcal{M}_a$, is given by $\mathcal{M}_a = \min(m_a) = 0.03$. This value is based on the results of computer simulations of a large number of bow arrow combinations, see [9].

There are also limits with respect to the real mass of the arrow. One has to bridge the distance between the middle of the string and the grip. To reduce the mass of the arrow

Table 7: An estimation of the nominal weight $\overline{\mathcal{F}}(\overline{\mathcal{D}})$ with $\overline{\mathcal{D}} = 0.7112$ m, of a number of types of bow for minimum arrow masses, that means $2\overline{m}_a = 2\overline{\mathcal{M}}_a = 0.01$ kg and $m_a = \mathcal{M}_a = 0.03$ so that $2\overline{m}_b = 0.33$ kg.

Material	Bow type	a_D	$\overline{\delta}_{bv}$ Nm/kg	A_b	$\overline{\mathcal{D}}\,\overline{\mathcal{F}}(\overline{\mathcal{D}})$ Nm	$\overline{\mathcal{F}}(\overline{\mathcal{D}})$ N
yew	KL	0.255	1100	0.52	180	260
sinew,horn,wood	PE	0.5	2000	0.59	564	800
sinew,horn,wood	TU	0.5	2000	1.10	303	420
glassfibre,maple	WR	0.33	3000	0.70	476	670

the Turk shot flight arrows shorter than the draw of the bow. This was made possible by using a horn groove ('siper') which they wore on the thumb or strapped to the wrist of the bow hand. Further, the arrow must be strong enough to withstand the acceleration force. We conclude that in practice there also seems to be a minimum, for the mass of the arrow. Suppose that this minimum, denoted as $2\overline{\mathcal{M}}_a$, is $2\overline{\mathcal{M}}_a = \min(2\overline{m}_a) = 0.01$ kg. This minimum might be a function of the weight of the bow when buckling of the arrow is the driving phenomena behind the constraint. So, in this case also the minimum value depends in a complex way on the other design parameters.

Combining both minimum values, the associated mass of the bow limbs becomes $2\overline{m}_b = 0.33$ kg. We assume that the product $a_D\,\overline{\delta}_{bv}$ is known. This value determines the maximum stored energy $\overline{A}_b$ by the use of Equation (24). For a yew bow with a D-shaped cross-section the value for this product equals about $0.255\,1100 = 280.5$ Nm/kg, see Tables 4 and 5. Therefore the maximum allowable stored energy $\max(\overline{A}_b)$ becomes about 93 Nm. For a straight-end KL-bow the dimensionless quantity $A_b = 0.52$ and therefore the product $|\overline{OD}|\,\overline{F}(|\overline{OD}|) = 180$ Nm. When the draw length is assumed to be $|\overline{OD}| = 0.7112$ m the maximum allowable weight of the bow becomes $\overline{F}(|\overline{OD}|) = 260$ N. We call this weight the *nominal weight* $\overline{\mathcal{F}}(\overline{\mathcal{D}})$ for a draw called the *nominal draw* $\overline{\mathcal{D}}$. So,

$$\overline{\mathcal{D}}\,\overline{\mathcal{F}}(\overline{\mathcal{D}}) = \frac{2\overline{\mathcal{M}}_a\,a_D\,\overline{\delta}_{bv}}{\mathcal{M}_a\,A_b}\,. \tag{30}$$

The nominal weight of a bow is fixed by its type (A_b) as well as by its construction ($a_D\,\overline{\delta}_{bv}$). The nominal draw $\overline{\mathcal{D}}$ is chosen equal to the 'span' of the archer.

Table 7 shows the data for each type of bow in combination with the materials of which they are commonly made. A value of 0.5 was chosen for a_D with the composite bow with sinew, horn and wood. This value is larger than 0.25 (the value for the elliptical cross-section) and takes into account the better usage of the material. The exact value is difficult to obtain as to do this the thickness of each layer must be known. The values for the quantity $\overline{\delta}_{bv}$ are combinations of those given in Table 4 for the materials used in making the bows. Those for A_b are taken from Table 3. We will use these results in Section 6 where we compare these nominal weights with data given in the literature.

For the bowyer it is now important to know whether the desired weight is larger or smaller than the nominal weight. We assume that the dimensionless parameters including $\mathcal{M}_a$ and also $2\overline{\mathcal{M}}_a$ together with the energy storage parameter $a_D\,\overline{\delta}_{bv}$ are already chosen. We will now discuss the influence of the weight $\overline{F}(|\overline{OD}|)$.

If $\overline{F}(|\overline{OD}|) > \overline{\mathcal{F}}(\overline{\mathcal{D}})$ then the bowyer must use more material and accordingly the mass of the limbs $\overline{m}_b$ becomes larger. This can be done by increasing the width of the limbs proportionally but leaving the thickness, as given by Equation (22), unchanged, but, the value for m_a, which is now a design parameter, was already as small as possible, so the archer has to use a heavier arrow ($\overline{m}_a > \overline{\mathcal{M}}_a$) when he wants to shoot such a strong bow. In this case all dimensionless design parameters remain the same and therefore he does not gain a larger initial velocity, but the kinetic energy increases proportionally with the weight, see Equations (26) and (27).

If $\overline{F}(|\overline{OD}|) < \overline{\mathcal{F}}(\overline{\mathcal{D}})$ the bowyer can remove some wood from the sides of the limbs. The mass of the arrow $\overline{m}_a$, which is now a design parameter, can not be lowered, and therefore the dimensionless mass of the arrow becomes larger ($m_a > \mathcal{M}_a$). This gives a better efficiency, but the negative effect of a smaller weight is stronger. For a given mass of the arrow the influence of the weight stems from the term $\eta\,\overline{F}(|\overline{OD}|)$ in Equation (27). The partial derivative of this term with respect to the weight of the bow equals, using Klopsteg's rule Equation (9) and Equation (17),

$$\frac{\partial(\eta\,\overline{F}(|\overline{OD}|))}{\partial\overline{F}(|\overline{OD}|)} = \eta + \overline{F}(|\overline{OD}|)\frac{\partial\eta}{\partial\overline{F}(|\overline{OD}|)} = \eta\,(1 - \frac{K_h}{m_a + K_h})\;. \tag{31}$$

Hence, initial velocity decreases for decreasing weight smaller than the nominal value.

These considerations show that for each combination of material and type of bow there is a nominal value for the product draw times weight which determines the maximum obtainable initial velocity (efficiency equal to 1) and it is given by Equations (26) and (27)

$$\max(\overline{c}_l) = \sqrt{q\frac{\overline{\mathcal{D}}\,\overline{\mathcal{F}}(\overline{\mathcal{D}})}{\overline{\mathcal{M}}_a}}\;. \tag{32}$$

5.2 Influence of the strength of material

When m_a is a design parameter ($\overline{F}(|\overline{OD}|) > \overline{\mathcal{F}}(\overline{\mathcal{D}})$) the advantage of a large value of the product $a_D\,\overline{\delta}_{bv}$ is clear from Equation (33). When, however, $\overline{m}_a$ is taken as a design parameter ($\overline{F}(|\overline{OD}|) < \overline{\mathcal{F}}(\overline{\mathcal{D}})$), the effect of a larger value of this product is neutralised by a larger value of m_a, better materials imply a lighter bow and therefore a larger m_a. The advantageous effect of better materials (larger $\overline{\delta}_{bv}$) or a better use of the materials (larger

a_D) stems in this case solely from the higher efficiency η. In the limit for $\overline{\delta}_{bv} \uparrow \infty$ we have $\overline{m}_b \downarrow 0$ and $m_a \uparrow \infty$, and this implies for the efficiency $\eta \uparrow 1$. In this case the drawn bow returns quasi-statically to the braced situation.

5.3 Influence of the static quality coefficient

In the literature a high value for the static quality coefficient q is often mentioned as a requirement of a good bow, see for instance [5–7]. Therefore it is interesting to study the influence of this quantity.

Substituting Klopsteg's rule, Equation (9) into Equation (25), we obtain for the velocity

$$\overline{c}_l = \sqrt{2 \frac{q}{A_b} \frac{1}{m_a + K_h} a_D \, \overline{\delta}_{bv}} \; . \tag{33}$$

When the dimensionless arrow mass m_a is taken as a design parameter ($\overline{F}(|\overline{OD}|) > \overline{\mathcal{F}}(\overline{\mathcal{D}})$), the first factor of the right-hand side of Equation (33) shows that the amount of energy A_H in braced situation, stored in the elastic parts of the bow, the limbs plus the string should be as small as possible. This can be explained as follows. The amount of energy A_D stored in the elastic parts in fully drawn situation is given by $A_D = A_b + A_s$. The data given in Table 2 for the KL-bow show that $A_H = 0.1226$, $A_b = 0.5155$, $A_s = 0.0137$ and $A_D = 0.5292$. The static quality coefficient q is given by $q = A_b + A_s - A_H$. This contemplation shows that a large q implies a large A_b.

Authors often stress the advantage of large recurve yielding a large static quality coefficient q, however, when the energy storage capacity in materials is taken into account, part of this advantage is offset by the relatively large energy contents of the braced bow. The results given in Table 2 show that for the TU-bow $A_H = 0.5959$ and for the ER-bow $A_H = 0.6160$ while for the KL-bow we have $A_H = 0.1226$.

When the mass of the arrow $\overline{m}_a$ is a design parameter ($\overline{F}(|\overline{OD}|) < \overline{\mathcal{F}}(\overline{\mathcal{D}})$), q and m_a in Equation (33) are mutually dependent, as a large q implies heavy limbs and this in turn yields a small m_a, because it is the arrow mass divided by the mass of one limb. In the case of a given mass of the arrow $\overline{m}_a$ and therefore fixed $\overline{d}_{av}$, the influence of q originates from the term $q\,\eta$ in Equation (27). We therefore calculate the partial derivative of this product with respect to q. The use of Klopsteg's rule Equation (9) and Equation (17) together with A_H assumed to be constant, gives

$$\frac{\partial(q\,\eta)}{\partial q} = \eta + q\frac{\partial \eta}{\partial q} = \eta(1 - \frac{q}{A_b}\frac{K_h}{m_a + K_h}) \; , \tag{34}$$

this quantity is always positive. So, we conclude that in this case a large static quality coefficient q is advantageous for both the initial velocity $\overline{c}_l$ and the kinetic energy $\overline{m}_a \overline{c}_l^2$.

The coefficient q is not a design parameter but a quality coefficient and thus a function of the dimensionless parameters which determine the static performance of the bow. Therefore we assumed in our derivation that it is possible to change the design parameters of Equation (1) so that the virtual mass K_h of the bow remains the same. In this way

Table 8: Parameters with dimension for a number of bows and an estimation of a_D, using Equation (17), and of the nominal weight $\overline{\mathcal{F}}(\overline{\mathcal{D}})$ with $\overline{\mathcal{D}} = |\overline{OD}|$.

Ref.	Bow type		$\overline{F}(\lvert\overline{OD}\rvert)$ N	$\lvert\overline{OD}\rvert$ m	$2\overline{m}_b$ kg	$2\overline{L}$ m	$\overline{\delta}_{bv}$ Nm/kg	A_b	a_D	$\overline{\mathcal{F}}(\overline{\mathcal{D}})$ N
[5]	flatbow	KL	155	0.7112	0.325	1.829	1100	0.52	0.20	198
[4]	longbow	KL	465	0.746	0.794	1.874	1100	0.52	0.25	236
[2]	steelbow	KL	170	0.7112	0.709	1.689	130	0.52	0.69	81
[16]	Tartar	PE	460	0.7366	1.47	1.88	2000	0.59	0.07	107
[15]	Turkish	TU	690	0.7112	0.35	1.14	2000	1.1	0.77	656
[18]	modern	WR	126	0.7112	0.29	1.703	3000	0.70	0.07	141

the influence of the static and dynamic coefficients on the initial velocity are separated completely. To manage this, it may be difficult to find a combination of adaptations of the design parameters. In reality such a combination does perhaps not exist, for instance, in the previous section we found that a larger q for a number of types of bow unfortunately was accompanied with a smaller efficiency. This holds for the TU-bow and certainly for the ER-bow as well.

The results obtained in this section show that the sensitivity of the velocity with respect to energy storage capacity and the static quality coefficient is different for $\overline{F}(|\overline{OD}|) < \overline{\mathcal{F}}(\overline{\mathcal{D}})$ and $\overline{F}(|\overline{OD}|) > \overline{\mathcal{F}}(\overline{\mathcal{D}})$. For $\overline{F}(|\overline{OD}|) = \overline{\mathcal{F}}(\overline{\mathcal{D}})$ the initial velocity is a maximum.

6 Comparison of a number of ancient and modern bows

In this section we consider a number of ancient bows described in the literature, [2,4,5,15,16] and one modern bow described in [18]. In Table 8 we give values for the parameters with dimension, weight, draw and mass of one limb. Estimations of a_D which have to be compared with the values given in Table 5 were calculated using Equation (17). The approximation of a_D has to be rough for lack of detailed information.

These results indicate that the short Turkish bow is made of a combination of good materials which are used to the full extent. The calculated value $a_D = 0.77$ is rather high, $\alpha_D = 0.25$ for an elliptic cross-section, see Table 5. Observe that the chosen value for $\overline{\delta}_{bv}$ is just the average of those for sinew and horn, for a better evaluation the thickness of the different layers should also be taken into account, but this detailed information is missing. If we take $a_D = 0.77$ instead of $a_D = 0.5$, used to calculate the value $\overline{\mathcal{F}}(\overline{\mathcal{D}}) = 420$ N mentioned in Table (7), we obtain $\overline{\mathcal{F}}(\overline{\mathcal{D}}) = 656$ N and the weight of the short Turkish bow is $\overline{F}(|\overline{OD}|) = 690$ N. So for this bow the weight is close to the nominal weight. The large amount of energy stored per unit of mass in the limbs explains the good performance of the Turkish bows in flight shooting, and not the mechanical performance of these bows; see Table 2.

The estimations of a_D for the flatbow and the longbow are very realistic. For an rectangular cross-section $\alpha_D = 0.33$ and for an D-shaped cross-section $\alpha_D = 0.255$, see Table 5. Note that the nominal weight for the flatbow is rather close to the weight of this bow. Indeed, Klopsteg designed this bow based on theoretical insight obtained by Hickman in [5]. For the steelbow, the estimation $a_D = 0.69$ is also close the value for $\alpha_D = 0.453$.

For both the Tartar bow and the modern the estimated value for $a_D = 0.07$. This indicates that not all the materials are used to its full extent. Observe that these bows are not used for flight shooting. For the Tartar bow the materials near the tips, namely that for the bridges, are stressed only partly. This results in a small efficiency.

For the modern working-recurve bow there is a surplus of material near the grip. An efficiency of 72.9% for the modern working-recurve bow is rather low. The maximum efficiency according to Equation (7) from the mass of the string determined by the parameters given in Table 3 is about 90%. This shows that with respect to this aspect, the design of the modern bow can be improved.

7 Conclusions

We conclude that the results indicate that the dimensionless initial velocity is about the same for all types of bow under similar conditions. So, within certain limits, the design parameters which determine the mechanical action of a bow-arrow combination appear to be less important than is often claimed.

Introduction of technological constraints and realistic assumptions about the cross-section of the limbs, yield a relationship between the distribution of the mass and bending stiffness along the limbs. This reduces the dimension of the design parameter space. It is advantageous to define the two nominal quantities of draw and weight. These depend on the type and construction of the bow. The sensitivity of the initial velocity with respect to design parameters is different depending on whether the weight is larger or smaller than the nominal weight. In the one case the dimensionless mass of the arrow and in the other the mass itself is a design parameter. The maximum initial velocity occurs when the weight of the bow equals the nominal weight for the type of bow considered. This maximum lies on the boundary of the feasible region with respect to the mass of the arrow.

It appears that the material properties of yew combine very well with the straight-end type of bow and the material properties of the composites, wood, sinew and horn combine very well with the static-recurve bow. A combination of many technical factors made the composite flight bow better suited for flight shooting.

After the recovery in 1982, of the *Mary Rose*, Henry's VIII's flagship which sank in the Solent in 1545, numerical modelling gave a good insight into the mechanical performance of the famous English longbow, used for instance at Agincourt during the Hundred Years' War between England and France, [17], and many other battles.

This shows that mathematical modelling is a helpful tool in the archery research, both for the design of new bow equipment and also for an understanding of the historical development of the bow.

The quality coefficients of the modern bow are only slightly better than those of the historical bows. Materials used in modern working-recurve bows can store more deformation energy per unit of mass than materials used in the past. Moreover the mechanical properties of these materials are more durable and much less sensitive to changing weather conditions. This contributes most to the improvement of the modern bow.

In this paper we have demonstrated the need to state the bow design problem in terms of structural optimization theory. Mathematical formulation helps with the systematic identification of all the design parameters. Sensitivity analysis yields the most important design parameters. Together with simplicity assumptions, this reduces the number of design parameters and makes the design process surveyable. This allows us to clarify the differences in the performance of bows found all over the world by the introduction of technological constraints.

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